Lanchester Models of the ARDENNES Campaign

Lanchester Models of the Ardennes Campaign

- Presented by
  - Muzaffer Coban, 1Lt, TUA
  - Sureyya Ardic, 1 Lt, TUA
  - Eng Yau Pee, DSTA
  - Wan Szu Ching, Maj, SAF

Past Studies (From Turkes's Thesis)

- Bracken, on the Ardennes campaign of World War II,
- Fricker, also on the Ardennes campaign,
- Clemens, on the Battle of Kursk of World War II,
- Hartley and Helmbold, on the Inchon-Seoul campaign of the Korean War
- Turkes, Fitting Lanchester and Other Equations to the Battle Of Kursk Data.

Lanchester Models of the Ardennes Campaign

- Detailed data base of the Ardennes campaign of World War II (December 15, 1944 through January 16, 1945) by Data Memory Systems, Inc.
- For US Army Concepts Analysis Agency.

Introduction

- Data : two sided , time-phased, and detailed.
- Another CAA data set. 600 battles and 140 different properties of each. (Good thesis opportunity with Prof. Lucas)

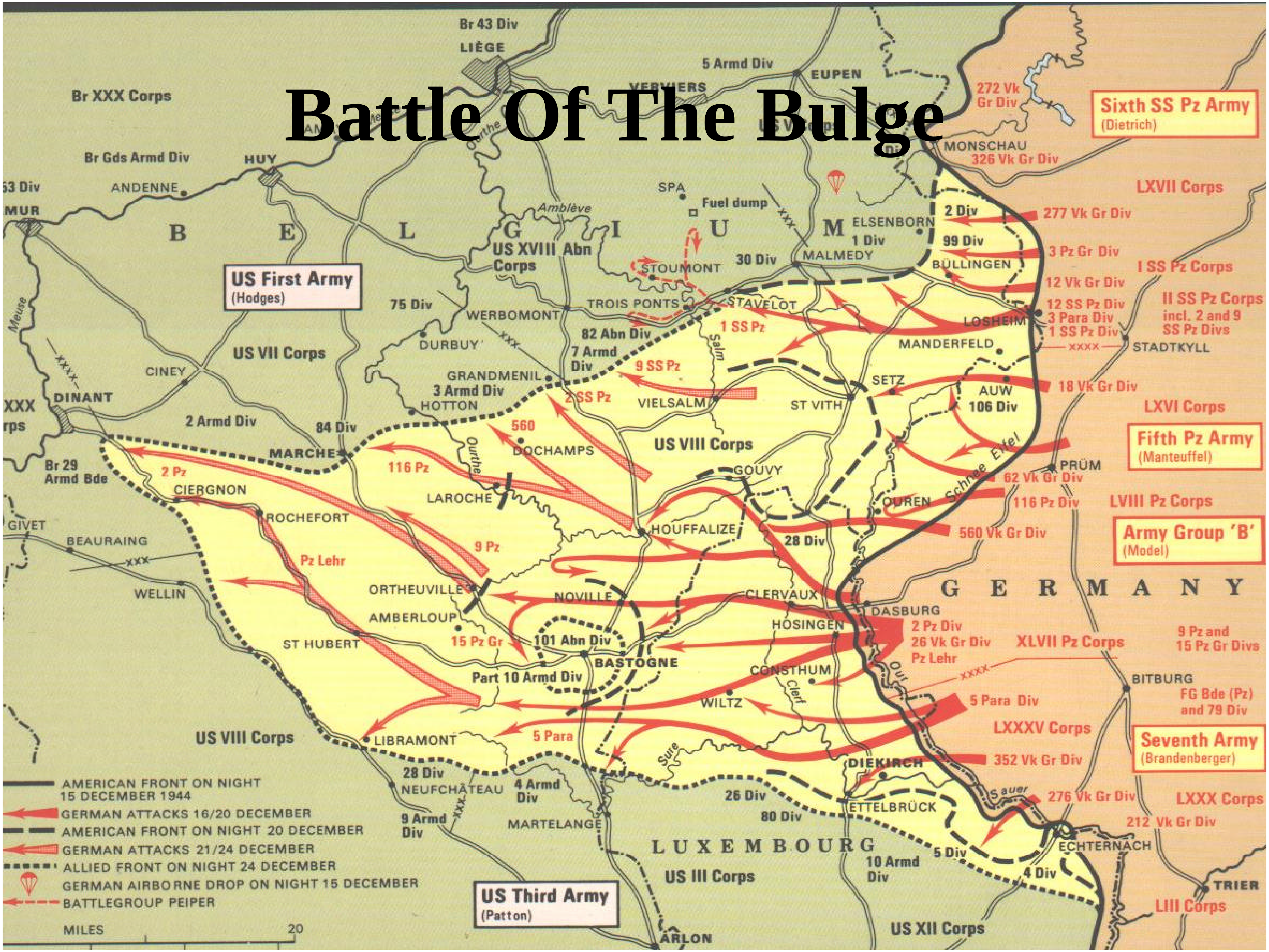
Introduction (Data)

- Data cover 33 days of the campaign from December 15, 1944 through January 16, 1945.
- The Germans attacked during days 1-6 and the Allies attacked during days 7-33.

Introduction (Data)

- The data of the first day is missing for the German side.
- The heaviest attrition takes place at the beginning of the campaign.
- The analysis treats the data for days 2-11. Five days during each side is attacking. 2-6 for Germans and 7-11 for allies.

Battle Of The Bulge



Historical Overview

- On December 16, 1944 three German armies launched a surprise attack against a thinly held section of the US front line on a stormy weather.
- Ardennes Campaign, known as Battle of the Bulge, caught US units by complete surprise.
- After several days of German penetrations, US forces slowed and then stopped the German attack

Historical Overview

- By Christmas day, the sky cleared and Allies counterattacked with the full might of the air supremacy.
- Approximately two weeks later, Allies restored the front line in the Ardennes.

Models Review

- General form of the model
- $\dot{B} = a(d \text{ or } 1/d)R^p B^q$
- $\dot{R} = b(1/d \text{ or } d)B^p R^q$
- B, R = blue forces, red forces
- $\dot{B}$, $\dot{R}$ = blue forces killed, red forces killed
- a,b = attrition parameters

Models Review (cont.)

- d,1/d = tactical parameter-factor for attrition to defender(d) or attrition to attacker (1/d)
- p = exponent parameter of shooting force
- q = exponent parameter of target force
- Daily data available for B and R
- Blue denotes Allied Forces, Red denotes the Germans

Models Review (cont.)

- $p=1$ and $q=0$ => Lanchester Square Law
- $p=1$ and $q=1$ => Lanchester Linear Law
- Square Law: $dx/dt = -ay$ and $dy/dt = -bx$
- Linear Law: $dx/dt = -axy$ and $dy/dt = -bxy$
- d is a multiplier of attrition due to being a defender

Models

With Tactical Parameters	Model 1 : Combat Forces	$\dot{B} = a(d \text{ or } 1/d)\, R^p B^q$ $\dot{R} = b(d \text{ or } 1/d)\, B^p R^q$
	Model 2 : Total Forces	$\dot{B} = a(d \text{ or } 1/d)\, R^p B^q$ $\dot{R} = b(d \text{ or } 1/d)\, B^p R^q$
Without Tactical Parameters	Model 3 : Combat Forces	$\dot{B} = a\, R^p B^q$ $\dot{R} = b\, B^p R^q$
	Model 4 : Total Forces	$\dot{B} = a\, R^p B^q$ $\dot{R} = b\, B^p R^q$

Discussion of Lanchester Models

- Hembold equation
- $dx/dt = -a(x/y)^{1-w}y$ and $dy/dt = -b(y/x)^{1-w}x$
- Models 1 and 2 have five parameters to be estimated whereas models 3 and 4 have four parameters to be estimated
- Parameters “a” and “b” are in Hembold’s general model

Discussion of Lanchester Models (cont.)

- Parameters p and q are estimated separately
- Parameter d is a bonus of the present analytical effort
- It significantly improves the fit
- Estimates are also made without “d” because it is not known in advance by force structure planners

Data On Tanks

Day	Blue Tanks	Blue tanks killed	Red Tanks	Red Tanks killed
1	2853	1	0	0
2	2863	12	747	10
3	2867	43	663	7
4	2840	60	639	13
5	2808	64	669	21
6	3965	33	619	11
7	4082	10	595	21

Data On Combat Manpower (Inf, Armour & Artillery)

Day	Blue manpower	Blue casualties	Red manpower	Red casualties
1	351005	458	0	0
2	349247	1589	360716	2191
3	347915	2383	356818	2423
4	358321	2085	353529	2015
5	366495	2175	350750	1993

Data On Combat Forces

Day	Blue forces	Blue losses	Red forces	Red losses
1	558820	478	1440	0
2	555482	2594	577446	2656
3	553625	3833	571923	4303
4	562661	3615	567134	3415
5	576795	4200	563255	3263

Notes On Data Tables

- Basic distinction between combat power and total manpower
- No distinction between surviving resources and newly arrived resources!!
- One avenue for research would be to attempt to estimate the weighting parameters rather than to assume them
- This is very difficult though

Estimation of Parameters

- Bracken’s technique (1995) of fitting Lanchester eqn. to Ardennes data
- ~~Justification of Bracken’s technique~~
- Fricker’s technique (1998) of fitting Lanchester eqn. to Ardennes data

Estimation of Parameters

$$\dot{B} = \frac{dB}{dt} = a\,(d \text{ or } {}^{1}\!/_{d})\,R^{p}B^{q}$$

$$\dot{R} = \frac{dR}{dt} = b\,({}^{1}\!/_{d} \text{ or } d)\,B^{p}R^{q}$$

- 5 parameters (*a*, *b*, *d*, *p*, *q*) to be estimated

Bracken's Technique

- Determine parameters by searching {a, b, p, q, d} grid space that minimizes residual sum-of-squares

$$SS = \sum_{n=2}^{6}\left(\dot{B}_n - a\,d\,R_n{}^{p}B_n{}^{q}\right)^2 + \sum_{n=2}^{6}\left(\dot{R}_n - b\,{}^{1}\!/_{d}\,B_n{}^{p}R_n{}^{q}\right)^2$$

$$+ \sum_{n=7}^{11}\left(\dot{B}_n - a\,{}^{1}\!/_{d}\,R_n{}^{p}B_n{}^{q}\right)^2 + \sum_{n=7}^{11}\left(\dot{R}_n - b\,d\,B_n{}^{p}R_n{}^{q}\right)^2$$

Parameter Grid (Model 1)

a	b	d	p	q
4×10^{-9}	4×10^{-9}	1	0.8	0.8
6×10^{-9}	6×10^{-9}	5/4	0.9	0.9
8×10^{-9}	8×10^{-9}	5/3	1.0	1.0
10×10^{-9}	10×10^{-9}		1.1	1.1
12×10^{-9}	12×10^{-9}		1.2	1.2

- $5\times5\times3\times5\times5 = 1{,}875$ combinations

Sum of Squared Residuals

Table 8. Sums of squared residuals for example.

$a_3 = 0.000000008$, $b_4 = .000000010$

	$q_1 = 0.8$	$q_2 = 0.9$	$q_3 = 1.0$	$q_4 = 1.1$	$q_5 = 1.2$	
$p_1 = 0.8$	0.273E+09	0.267E+09	0.244E+09	0.167E+09	0.259E+08	$d_1 = 10/10$
$p_2 = 0.9$	0.266E+09	0.243E+09	0.167E+09	0.247E+08	0.161E+10	
$p_3 = 1.0$	0.243E+09	0.166E+09	0.236E+08	0.162E+10	0.382E+11	
$p_4 = 1.1$	0.166E+09	0.226E+08	0.163E+10	0.384E+11	0.616E+12	
$p_5 = 1.2$	0.217E+08	0.164E+10	0.386E+11	0.620E+12	0.913E+13	
	0.273E+09	0.266E+09	0.241E+09	0.160E+09	0.166E+08	$d_2 = 10/8$
	0.266E+09	0.241E+09	0.160E+09	0.164E+08	0.182E+10	
	0.241E+09	0.159E+09	$\boxed{0.163E+08}$	0.186E+10	0.427E+11	
	0.158E+09	0.164E+08	0.189E+10	0.433E+11	0.690E+12	
	0.165E+08	0.193E+10	0.440E+11	0.699E+12	0.103E+14	
	0.272E+09	0.264E+09	0.236E+09	0.147E+09	0.405E+08	$d_3 = 10/6$
	0.264E+09	0.236E+09	0.146E+09	0.423E+08	0.301E+10	
	0.236E+09	0.145E+09	0.443E+08	0.308E+10	0.632E+11	
	0.144E+09	0.464E+08	0.317E+10	0.646E+11	0.100E+13	
	0.488E+08	0.325E+10	0.660E+11	0.102E+13	0.149E+14	

Mean daily attrition: 0.7027E+04 = 7027. Standard deviation = $\sqrt{0.1633E+08/10} = 1278$.

Best Fit Models

Model	*a*	*b*	*d*	*p*	*q*
1	8×10^{-9}	10×10^{-9}	1.25	1.0	1.0
2	8×10^{-9}	8×10^{-9}	1.25	0.8	1.2
3	8×10^{-9}	10×10^{-9}	-	1.3	0.7
4	8×10^{-9}	8×10^{-9}	-	1.2	0.8

Shortfalls in Bracken's Technique

- Bracken: "…does not guarantee that an optimal fit be found"
- Parameter grid derived through experience and trial & error
- Not exhaustive search

Fricker’s Technique

- Ronald D Fricker, Jr - RAND, Santa Monica, CA
- Recommended by Bracken to undertake the analysis
- “Attrition models of the Ardennes campaign”, Naval Research Logistics, Vol 45, no.1, 1998
- Same 4 models

Fricker's Technique

- Differences with Bracken:
  - Linear regression applied
  - Use data from entire campaign, i.e. day 2-33
  - Includes air sortie data, each sortie weighted at 30

Fricker's Technique

$$\dot{B} = \frac{dB}{dt} = a\,(d \text{ or } {}^{1}\!/_{d})\, R^{p} B^{q}$$

$$\dot{R} = \frac{dR}{dt} = b\,({}^{1}\!/_{d} \text{ or } d)\, B^{p} R^{q}$$

ß

$$\log(\dot{B}) = \log(a) + \log\,(d \text{ or } {}^{1}\!/_{d}) + p\log(R) + q\log(B)$$

$$\log(\dot{R}) = \log(b) + \log\,({}^{1}\!/_{d} \text{ or } d) + p\log(B) + q\log(R)$$

Fricker's Technique

- Advantages:
  - SS is minimized
  - Statistical techniques can be used to judge the significance of the parameters and the fit of the model

Results

With Tactical Parameters	Model 1 : Combat Forces	$\dot{B} = 0.000\,000\,008(\frac{10}{8} \text{ or } \frac{8}{10})\, R^{1}\, B^{1}$ $\dot{R} = 0.000\,000\,010(\frac{8}{10} \text{ or } \frac{10}{8})\, B^{1}\, R^{1}$
	Model 2 : Total Forces	$\dot{B} = 0.000\,000\,008(\frac{10}{8} \text{ or } \frac{8}{10})\, R^{0.8}\, B^{1.2}$ $\dot{R} = 0.000\,000\,008(\frac{8}{10} \text{ or } \frac{10}{8})\, B^{0.8}\, R^{1.2}$
Without Tactical Parameters	Model 3 : Combat Forces	$\dot{B} = 0.000\,000\,008\, R^{1.3}\, B^{0.7}$ $\dot{R} = 0.000\,000\,010\, B^{1.3}\, R^{0.7}$
	Model 4 : Total Forces	$\dot{B} = 0.000\,000\,008\, R^{1.2}\, B^{0.8}$ $\dot{R} = 0.000\,000\,008\, B^{1.2}\, R^{0.8}$

Results

Table 9. Model 1—Sums of squared residuals and NRL-804 actuals, estimates, and residuals for best fit.

	Sums of squared residuals					
	$a_3 = 0.000000008$ $b_4 = 0.000000010$					
	$q_1 = 0.8$	$q_2 = 0.9$	$q_3 = 1.0$	$q_4 = 1.1$	$q_5 = 1.2$	
$p_1 = 0.8$	0.273E+09	0.267E+09	0.244E+09	0.167E+09	0.259E+08	$d_1 = 10/10$
$p_2 = 0.9$	0.266E+09	0.243E+09	0.167E+09	0.247E+08	0.161E+10	
$p_3 = 1.0$	0.243E+09	0.166E+09	0.236E+08	0.162E+10	0.382E+11	
$p_4 = 1.1$	0.166E+09	0.226E+08	0.163E+10	0.384E+11	0.616E+12	
$p_5 = 1.2$	0.217E+08	0.164E+10	0.386E+11	0.620E+12	0.913E+13	
	0.273E+09	0.266E+09	0.241E+09	0.160E+09	0.166E+08	$d_2 = 10/8$
	0.266E+09	0.241E+09	0.160E+09	0.164E+08	0.182E+10	
	0.241E+09	0.159E+09	$\boxed{0.163E+08}$	0.186E+10	0.427E+11	
	0.158E+09	0.164E+08	0.189E+10	0.433E+11	0.690E+12	
	0.165E+08	0.193E+10	0.440E+11	0.699E+12	0.103E+14	
	0.272E+09	0.264E+09	0.236E+09	0.147E+09	0.405E+08	$d_3 = 10/6$
	0.264E+09	0.236E+09	0.146E+09	0.423E+08	0.301E+10	
	0.236E+09	0.145E+09	0.443E+08	0.308E+10	0.632E+11	
	0.144E+09	0.464E+08	0.317E+10	0.646E+11	0.100E+13	
	0.488E+08	0.325E+10	0.660E+11	0.102E+13	0.149E+14	

Table 9 : Model 1 (pg 430)

Day	Blue losses	Est blue losses	Residual (Blue)	Red losses	Est Red losses	Residual (Red)
2	2594	3208	*- 614*	2656	2566	90
3	3833	3166	667	4303	2533	1770
4	3615	3191	424	3415	2553	862
5	4200	3249	951	3263	2599	664
6	3424	3672	*- 248*	3275	2938	337
7	1804	2415	*- 611*	3799	4718	*- 919*
8	2350	2523	*- 173*	2866	4929	*- 2063*
9	2698	2519	179	4518	4920	*- 402*
10	2858	2595	263	6985	5068	1917
11	2177	2609	*- 432*	5638	5096	542

Average total losses = 7027. Standard deviation = 1278.

Interpretation of Results

- ***First interpretation*** : Lanchester linear equation fits the campaign

Results

Table 9. Model 1—Sums of squared residuals and NRL-804 actuals, estimates, and residuals for best fit.

	Sums of squared residuals					
	$a_3 = 0.000000008$ $b_4 = 0.000000010$					
	$q_1 = 0.8$	$q_2 = 0.9$	$q_3 = 1.0$	$q_4 = 1.1$	$q_5 = 1.2$	
$p_1 = 0.8$	0.273E+09	0.267E+09	0.244E+09	0.167E+09	0.259E+08	$d_1 = 10/10$
$p_2 = 0.9$	0.266E+09	0.243E+09	0.167E+09	0.247E+08	0.161E+10	
$p_3 = 1.0$	0.243E+09	0.166E+09	0.236E+08	0.162E+10	0.382E+11	
$p_4 = 1.1$	0.166E+09	0.226E+08	0.163E+10	0.384E+11	0.616E+12	
$p_5 = 1.2$	0.217E+08	0.164E+10	0.386E+11	0.620E+12	0.913E+13	
	0.273E+09	0.266E+09	0.241E+09	0.160E+09	0.166E+08	$d_2 = 10/8$
	0.266E+09	0.241E+09	0.160E+09	0.164E+08	0.182E+10	
	0.241E+09	0.159E+09	0.163E+08	0.186E+10	0.427E+11	
	0.158E+09	0.164E+08	0.189E+10	0.433E+11	0.690E+12	
	0.165E+08	0.193E+10	0.440E+11	0.699E+12	0.103E+14	
	0.272E+09	0.264E+09	0.236E+09	0.147E+09	0.405E+08	$d_3 = 10/6$
	0.264E+09	0.236E+09	0.146E+09	0.423E+08	0.301E+10	
	0.236E+09	0.145E+09	0.443E+08	0.308E+10	0.632E+11	
	0.144E+09	0.464E+08	0.317E+10	0.646E+11	0.100E+13	
	0.488E+08	0.325E+10	0.660E+11	0.102E+13	0.149E+14	

Results

Table 10. Model 2—Sums of squared residuals and actuals, estimates, and residuals for best fit.

Sums of squared residuals

$a_3 = 0.000000008 \quad b_3 = 0.00000008$

	$q_1 = 0.6$	$q_2 = 0.8$	$q_3 = 1.0$	$q_4 = 1.2$	$q_5 = 1.4$	
$p_1 = 0.6$	0.965E+09	0.965E+09	0.958E+09	0.858E+09	0.159E+09	$d_1 = 10/10$
$p_2 = 0.8$	0.965E+09	0.958E+09	0.858E+09	0.137E+09	0.192E+12	
$p_3 = 1.0$	0.958E+09	0.858E+09	0.124E+09	0.189E+12	0.520E+14	
$p_4 = 1.2$	0.857E+09	0.119E+09	0.188E+12	0.516E+14	0.127E+17	
$p_5 = 1.4$	0.123E+09	0.192E+12	0.520E+14	0.127E+17	0.308E+19	
	0.965E+09	0.964E+09	0.957E+09	0.852E+09	0.990E+08	$d_2 = 10/8$
	0.964E+09	0.957E+09	0.851E+09	0.938E+08	0.203E+12	
	0.957E+09	0.850E+09	0.974E+08	0.205E+12	0.557E+14	
	0.849E+09	0.110E+09	0.211E+12	0.568E+14	0.137E+17	
	0.133E+09	0.222E+12	0.589E+14	0.141E+17	0.339E+19	
	0.965E+09	0.964E+09	0.956E+09	0.836E+09	0.176E+09	$d_3 = 10/6$
	0.964E+09	0.956E+09	0.835E+09	0.193E+09	0.285E+12	
	0.956E+09	0.833E+09	0.223E+09	0.296E+12	0.775E+14	
	0.831E+09	0.268E+09	0.312E+12	0.810E+14	0.193E+17	
	0.329E+09	0.334E+12	0.859E+14	0.204E+17	0.483E+19	

Results

Table 11. Model 3—Sums of squared residuals and actuals, estimates, and residuals for best fit.

	Sums of squared residuals				
	$a_3 = 0.000000008$ $b_4 = 0.000000010$				
	$q_1 = 0.4$	$q_2 = 0.7$	$q_3 = 1.0$	$q_4 = 1.3$	$q_5 = 1.7$
$p_1 = 0.4$	0.275E+09	0.275E+09	0.275E+09	0.267E+09	0.162E+10
$p_2 = 0.7$	0.275E+09	0.275E+09	0.267E+09	0.272E+08	0.899E+13
$p_3 = 1.0$	0.275E+09	0.266E+09	0.236E+08	0.614E+12	0.269E+17
$p_4 = 1.3$	0.266E+09	0.208E+08	0.623E+12	0.190E+16	0.799E+20
$p_5 = 1.7$	0.172E+10	0.947E+13	0.279E+17	0.816E+20	0.343E+25

Interpretation of Results

- ***First interpretation*** : Lanchester linear equation fits the campaign
- ***Second interpretation*** : the individual effectiveness parameters depend upon whether combat forces in the campaign or total forces in the campaign are included.

Results

With Tactical Parameters	Model 1 : Combat Forces	$\dot{B} = 0.000\,000\,008(\frac{10}{8} \text{ or } \frac{8}{10})\, R^{1}\, B^{1}$ $\dot{R} = 0.000\,000\,010(\frac{8}{10} \text{ or } \frac{10}{8})\, B^{1}\, R^{1}$
	Model 2 : Total Forces	$\dot{B} = 0.000\,000\,008(\frac{10}{8} \text{ or } \frac{8}{10})\, R^{0.8}\, B^{1.2}$ $\dot{R} = 0.000\,000\,008(\frac{8}{10} \text{ or } \frac{10}{8})\, B^{0.8}\, R^{1.2}$
Without Tactical Parameters	Model 3 : Combat Forces	$\dot{B} = 0.000\,000\,008\, R^{1.3}\, B^{0.7}$ $\dot{R} = 0.000\,000\,010\, B^{1.3}\, R^{0.7}$
	Model 4 : Total Forces	$\dot{B} = 0.000\,000\,008\, R^{1.2}\, B^{0.8}$ $\dot{R} = 0.000\,000\,008\, B^{1.2}\, R^{0.8}$

Limitations of Studies

- Models used in the studies are homogeneous in the sense that reasonable but ***subjective*** weights are assigned to the combat elements to estimate the parameters
  - Alternative : To use heterogeneous models; but would would involve many more parameters. Similar approach to the study of the American Civil War battles could be adopted for the study of Ardennes campaign

Limitations of Studies (cont.)

- Strictly speaking, Lanchester equations only represent the combat forces physically in engagement. But the non-combat elements were used in the events where the total forces are included, i.e.theory and empirical work do not strictly correspond
  - This area of model definition and scope might be useful for further investigation

Limitations of Studies (cont.)

- Recall : Estimation of parameters based on the range of 5 values each for a, b, p, q, and 3 values for d, a total of 1875 variations
- Following are not explored:
  - Detailed variations in parameters to obtained best fits or at least tighter fits
  - Presence and effects of local minimal

Limitations of Studies (cont.)

- Finally, effects of air battles not examined

Conclusions

- A good start point to validate Lanchester models against data from a 2-sided time histories of warfare on battles
- Lanchester Linear Law fits all 4 models used
- Also showed that the individual effectiveness of 2 fighting forces can be identical, despite their different organizational configuration
- Some scope for further studies on the limitations of the Ardennes campaign study

Question ?

Question 1

- In Bracken’s study, he concluded that the Lanchester Square Law fitted the Ardennes data. (T/F)

Question 2

- The tactical parameter “d” in Bracken’s models accounts for attacker/defender advantage. (T/F)

Question 3

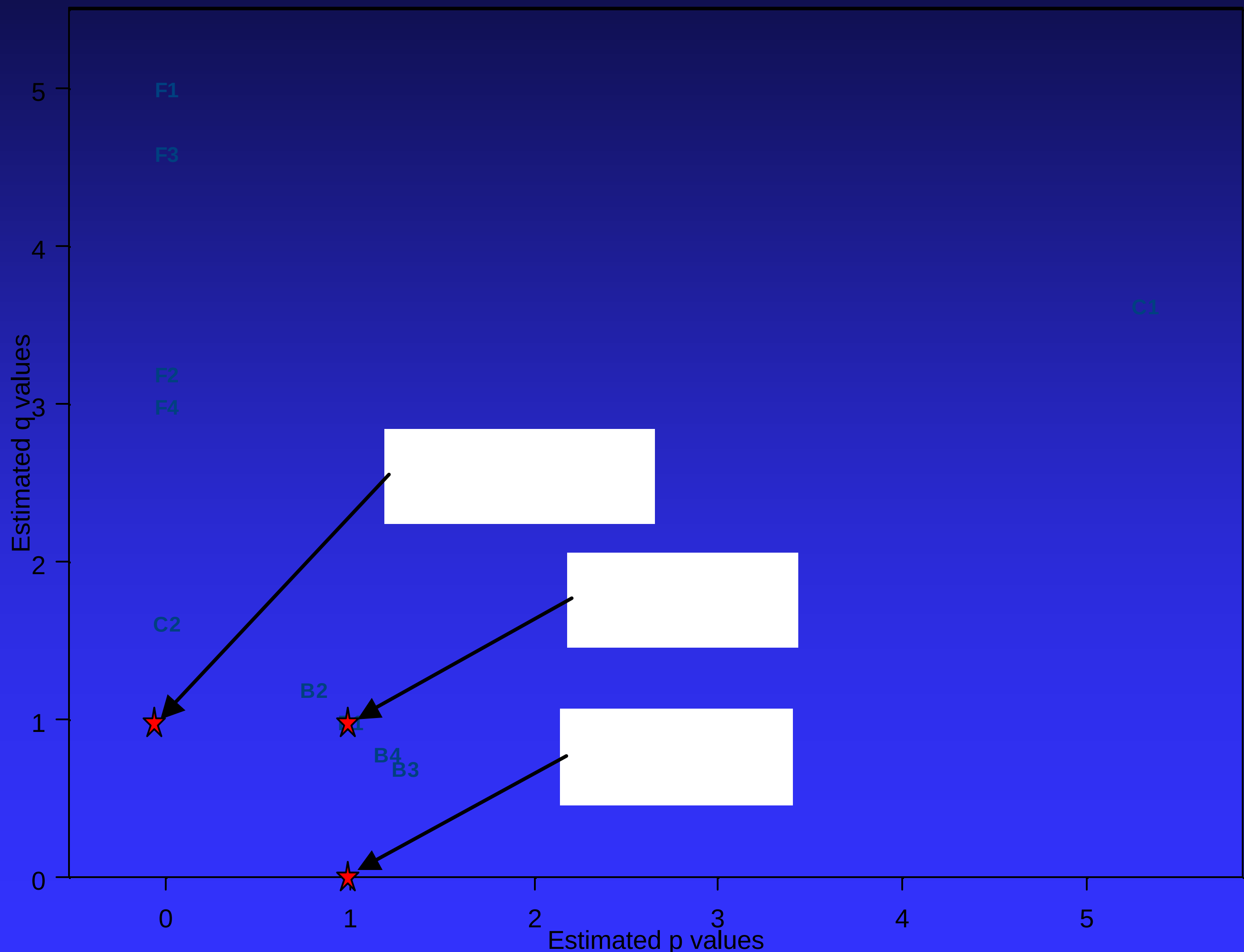
- What are the limitations of the Ardennes campaign study?
  - Models used are homogeneous in which reasonable but subjective weights are assigned to the combat elements
  - Theory and empirical work do not strictly correspond, in that the non-combat elements are included in the estimation of parameters in Lanchester equation, which strictly speaking only represents combat forces in physical engagement
  - Detailed variations in parameters to obtained best fits or at least tighter fits, and the presence and effects of local minimal not thoroughly explored
  - Effects of air battles not examined

Bracken Follow-up

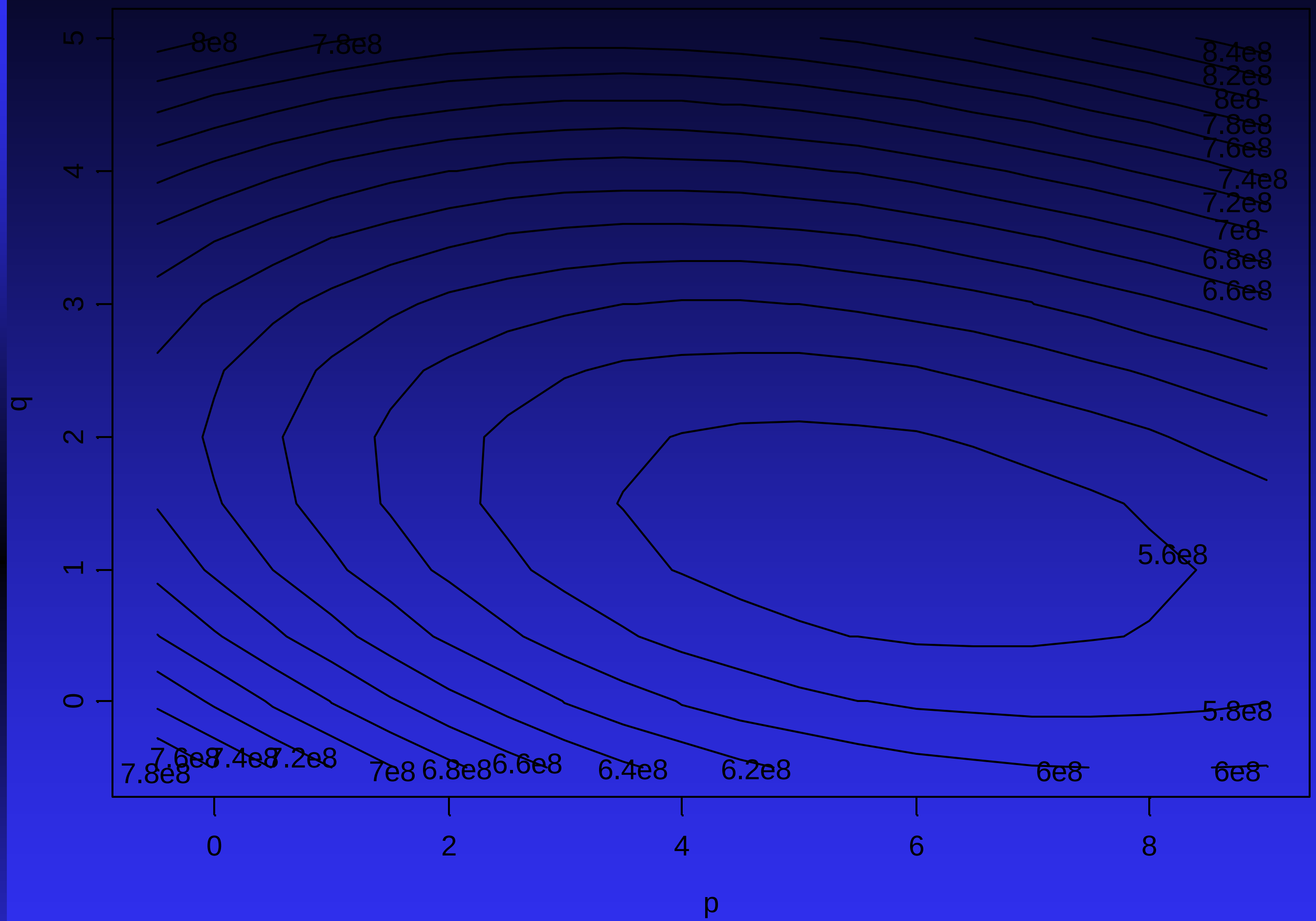
Follow-on Research

- Fricker
  - Use regression on logarithmically transformed versions of Bracken's generalized Lanchester equations
  - All 32 Days, Air Sorties
  - RESULT: P = 0.0, Q = 4.6
- Clemens
  - Used Kursk Data
  - Regression + Newton-Raphson
  - Newton-Raphson: $(p, q) = (0, 1.62)$, regression $(p, q) = (5.32, 3.63)$
- Turker (NPS Thesis)

Plot of Cumulative Findings



A Better Approach: The Kursk Surface



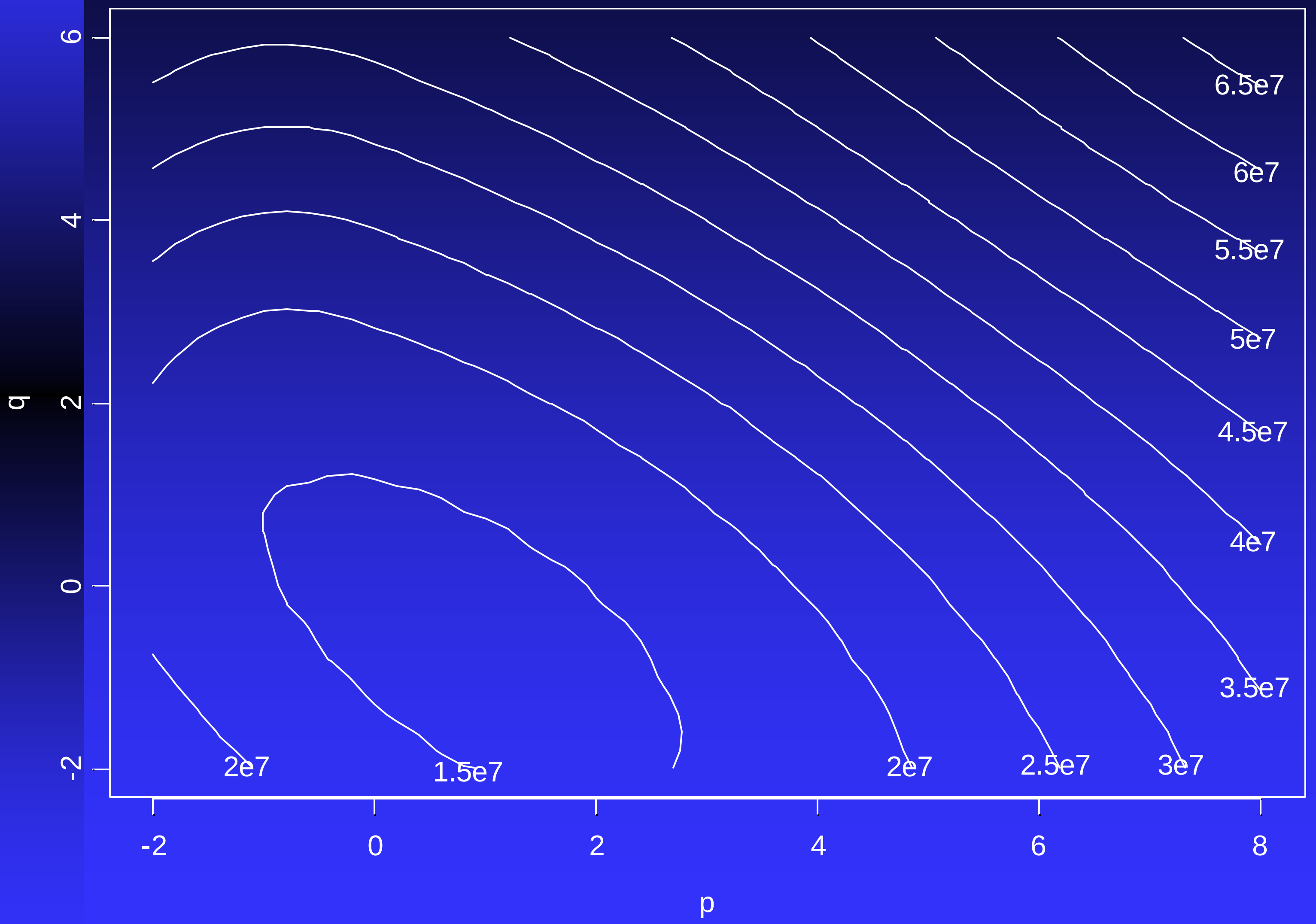
$$\dot{B} = 1.4658 \times 10^{-35} R^{5.6957} B^{1.2702}$$

$$\dot{R} = 1.2014 \times 10^{-36} B^{5.6957} R^{1.2702}$$

$R^2 = 0.237$

*** With *d* (= 1.028) $R^2 = 0.238$**

The Ardennes Surface



Some of Turker's Conclusions

- Constant attrition homogeneous Lanchester equations don't seem to fit the Kursk data well
  - Linear best of the basic
- Kursk and Ardennes give different best fitting models/surfaces
- Response surface is fairly flat over broad regions (far from the basic models)
- Change points dramatically improve fit
- Results seem relatively insensitive to weights (more to be done)
- Co-linearity adversely affects estimation
- Results can be sensitive to how the data is formatted
- No clear defender advantage (if anything a slight attacker advantage)
- Inclusion of Air Sorties does not improve the fit